AN INDUCTION MACHINE MODEL BASED ON ANALYTIC TWO-DIMENSIONAL FIELD COMPUTATIONS

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Abstract
This paper presents a coherent description of an induction machine in which two-dimensional analytical field analysis, a network description of the magnetic circuit, and the classical equivalent circuit are combined. In this description, classical machine parameters (main and leakage inductances) and quantities like the main and leakage fluxes take on a meaning for the case of a large air gap (two-dimensional field analysis).
This description can be used to simplify two-dimensional field computations for creating field plots, which are of great value for teaching electrical machines.

1. - INTRODUCTION

One of the first suppositions made in teaching electrical machines is that the air gap is small compared to its radius. In this way the air-gap field computation becomes very easy because the field only has a radial component. However with only a radial field, the stator cannot exert a torque on the rotor. This contradiction is solved by supposing that the force is exerted on currents in windings in the air gap (neglecting their tangential field).
I would like to show that it is possible to use a quite simple analytic two-dimensional field analysis to show the bending of the field lines in the air gap corresponding to the torque. Besides, the air-gap leakage flux is made visible too. Because the computation of the field plot is very fast, it can also be used to illustrate phenomena described in, for example, [4].
It is not my intention to show the students the two-dimensional field analysis, although it is not very complicated. I just want to have a clear model to make computer animations for showing different phenomena in induction machines. By making the air gap relative large, these phenomena become very clear. This also holds for the air-gap leakage flux.
The result of this article is a coherent description of an induction machine in which two-dimensional analytical field analysis, a network description of the magnetic circuit, and the classical equivalent circuit are combined. In this description, classical machine parameters (main and leakage inductances) and quantities like the main and leakage fluxes take on a meaning for the case of a large air gap (two-dimensional field analysis). Further, the magnetic equivalent circuit can be used to simplify two-dimensional field computations for creating field plots.
We start with a physical model of an induction machine. Next, the magnetic field is computed analytically. We use the expressions found to derive a magnetic network and the corresponding equivalent circuit. In section 7, the flux linkages from the equivalent circuit are used in the rotor and stator voltage equations. In this way we have found the equations corresponding to the model of the induction machine. In section 8 some examples are given.

2. - THE PHYSICAL MACHINE MODEL

The machine model is based on the usual assumptions for modelling induction machines, except that the air-gap length doesn't have to be infinitely small (figure 1).
The stator (*s*) winding consists of three (*a*, *b*, and *c*) sinusoidally distributed, infinitely thin windings along the stator circumference in the air gap, the rotor (*r*) winding of three such windings along the rotor circumference. The

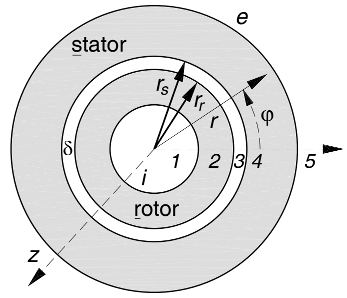

Figure 1 A representation of the induction machine

magnetic axes of these windings are depicted in figure 2 for a 2-pole machine. The angle θ is the rotor position angle.

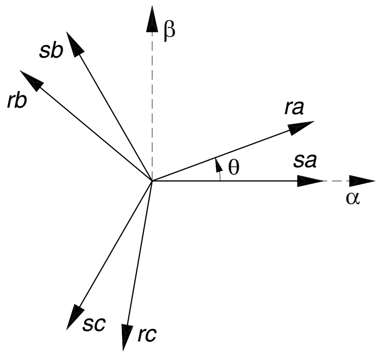

Figure 2 The magnetic axes of a 2-pole induction machine

As an example, we describe the winding of stator phase a, the winding distribution (number of conductors per metre) of which is expressed as

$$Z_{sa}(\varphi) = \hat{Z}_s \sin p\varphi \tag{1}$$

where p is the number of pole pairs and φ is the angle of the cylindrical coordinate system which is used (see the figures 1 and 2). All conductors of one phase are supposed to be connected in series. Hence, the total number of turns of a phase winding is $N_s = 2r_s\hat{Z}_s$ and we can write the winding distribution (1) as

$$Z_{sa}(\varphi) = \frac{N_s}{2r_s}\sin p\varphi \tag{2}$$

and the corresponding surface current density as

$$K_{sa}(\varphi) = i_{sa}\hat{Z}_s \sin p\varphi = \frac{N_s}{2r_s} i_{sa} \sin p\varphi \tag{3}$$

3. - THE MAGNETIC FIELD

First, the magnetic field is computed analytically. The two-dimensional field analysis in cylindrical coordinates used is quite old already. In [2] was mentioned that its text is a reprint of a text originally published in 1929.

For the analysis of the magnetic field, we only consider the case in which the magnetization direction is along the horizontal axis. Later, we can rotate this axis, which results in a rotation of the whole field pattern.

3.1 - The field equation in cylindrical coordinates

This subsection describes how to combine (Maxwell's) equations to obtain one partial differential equation for each region in figure 1, from which the magnetic field in that region can be solved. A more comprehensive description may be found in many text books, for example, [1] and [3].

Because we assumed the windings to be infinitely thin on the borders of the air gap, the current density is zero in a region and Ampere's law is given by

$$\nabla \times \vec{H} = 0 \tag{4}$$

where $\vec{H}$ is the magnetic field strength.

The relation between the magnetic flux density $\vec{B}$ and $\vec{H}$ is given by $\vec{B} = \mu\vec{H}$, where μ is the magnetic permeability, which is supposed ot be constant in the region. Using this in equation (4) results in

$$\nabla \times \vec{B} = \mu \nabla \times \vec{H} = 0 \tag{5}$$

To simplify the equations and the creation of field plots, we use the magnetic vector potential $\vec{A}$: $\vec{B} = \nabla \times \vec{A}$. Substitution of this expression in equation (5) results in:

$$\nabla \times (\nabla \times \vec{A}) = 0 \tag{6}$$

Using the gauge $\nabla \cdot \vec{A} = 0$, this may be written as

$$\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2\vec{A} = -\nabla^2\vec{A} = 0 \tag{7}$$

Here, the field quantities are considered to be independent of z (two-dimensional field analysis). Therefore, (7) can be written as (in cylindrical coordinates)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 A_z}{\partial \varphi^2} = 0 \tag{8}$$

We may solve this equation by using the method of separation of variables:

$$\begin{aligned} A_z(r,\varphi) = \sum_{k=1}^{\infty} \Big\{ &(C_{\alpha,k}r^{-k} + D_{\alpha,k}r^k)\sin k\varphi \\ &-(C_{\beta,k}r^{-k} + D_{\beta,k}r^k)\cos k\varphi \Big\} \end{aligned} \tag{9}$$

The constants $C_{\alpha,k}$, $D_{\alpha,k}$, $C_{\beta,k}$, and $D_{\beta,k}$ are determined by the boundary conditions. Because we only consider the case in which the magnetization direction is along the horizontal axis, the vector potential is mirror symmetrical with respect to the vertical axis.

As we may see in expression (9), this also means that the coefficients with subscript β are zero. Therefore, we don't use the subscript α from now on.

Further, the vector potential linearly depends on its excitation, the surface current densities. Because they only contain sine waves with $\sin p\varphi$, we only use the term in the Fourier series with $k = p$.

Besides, we divided the space in cylindrical regions (see figure 1), which are indicated by the subscript m (here, $m = 1...5$):

$$A_{z,m}(r,\varphi) = \tilde{A}_{z,m}(r)\sin p\varphi \tag{10}$$

where

$$\tilde{A}_{z,m}(r) = C_m r^{-p} + D_m r^p \tag{11}$$

The function $\tilde{A}_{z,m}(r)$ may be seen as a kind of amplitude. However, it may have a negative value.

The radial flux density and the tangential magnetic field strength can be expressed as

$$B_{r,m}(r,\varphi) = \frac{1}{r}\frac{\partial A_{z,m}}{\partial \varphi} = \frac{p}{r}\tilde{A}_{z,m}(r)\cos p\varphi \tag{12}$$

$$H_{\varphi,m}(r,\varphi) = -\frac{1}{\mu_m}\frac{\partial A_{z,m}}{\partial r} = \tilde{H}_{\varphi,m}(r)\sin p\varphi \tag{13}$$

where

$$\tilde{H}_{\varphi,m}(r) = -\frac{1}{\mu_m}\frac{\mathrm{d}\tilde{A}_{z,m}(r)}{\mathrm{d}r} \tag{14}$$

Because all functions are sinusoidal and represent one magnetization direction, we only need the functions $\tilde{A}_{z,m}(r)$ and $\tilde{H}_{\varphi,m}(r)$.

3.2 - **The boundary conditions**

We can solve the set of equations by means of the boundary conditions. The first is:

$$B_{r,m}(r_{o,m},\varphi) = B_{r,m+1}(r_{i,m+1},\varphi)$$

where $r_{o,m}$ is the radius of the outer border of region m and $r_{i,m+1}$ is the radius of the inner border of region $m+1$, which are equal of course ($r_{o,m} = r_{i,m+1}$).
When we look at equation (12), we can see that we only need the equation for $\tilde{A}_z$:

$$\tilde{A}_{z,o,m} = \tilde{A}_{z,i,m+1} \tag{15}$$

where $\tilde{A}_{z,o,m}$ is the value of $\tilde{A}_z$ at the outer border of region m and $\tilde{A}_{z,i,m+1}$ this value at the inner border of region $m+1$.
The second boundary condition is related to the tangential component of the magnetic field strength. Using (3) for the surface current density, we get:

$$H_{\varphi,m}(r_m,\varphi) - H_{\varphi,m+1}(r_m,\varphi) = -K_m(\varphi) = -i_m\hat{Z}_m \sin p\varphi$$

Here, we also only need the equation for $\tilde{H}_\varphi$:

$$-\tilde{H}_{\varphi,o,m} + \tilde{H}_{\varphi,i,m+1} = i_m\hat{Z}_m \tag{16}$$

where $\tilde{H}_{\varphi,o,m}$ is the value of $\tilde{H}_\varphi$ at the outer border of region m and $\tilde{H}_{\varphi,i,m+1}$ this value at the inner border of region $m+1$.

3.3 - **New forms for the vector potential**

We can directly find the coefficients C and D in the expression for $\tilde{A}_z$ ((11)) by means of the boundary conditions (15) and (16) and the characteristic equation for the region (14). However, these coefficients have no physical meaning and it may be useful to express these coefficients as functions of two other quantities, especially, when a combination of quantities corresponding with the boundary conditions is used. Here, we choose the values of the magnetic field strengths (using (14) and (11)):

$$\begin{aligned}\tilde{H}_{\varphi,i} &= \tilde{H}_\varphi(r_i) = \frac{p}{\mu r_i}\left(Cr_i^{-p} - Dr_i^{p}\right)\\ \tilde{H}_{\varphi,o} &= \tilde{H}_\varphi(r_o) = \frac{p}{\mu r_o}\left(Cr_o^{-p} - Dr_o^{p}\right)\end{aligned} \tag{17}$$

where i stands for inner and o for outer border. Because we are only considering one of the regions in this subsection, we may leave out the subscript m.
From these equations we can find expressions for C and D. Using these, we can write for the vector potential ((11)):

$$\tilde{A}_z(r) = \frac{\mu k\varepsilon}{2p}\left(r_i\tilde{H}_{\varphi,i}\left[\frac{r_o^p}{r^p} + \frac{r^p}{r_o^p}\right] - r_o\tilde{H}_{\varphi,o}\left[\frac{r^p}{r_i^p} + \frac{r_i^p}{r^p}\right]\right) \tag{18}$$

where we used the parameters ε and k according to

$$\varepsilon = \frac{\frac{r_o^p}{r_i^p} + \frac{r_i^p}{r_o^p}}{\frac{r_o^p}{r_i^p} - \frac{r_i^p}{r_o^p}} \quad ; \quad k = \frac{2}{\frac{r_o^p}{r_i^p} + \frac{r_i^p}{r_o^p}} \tag{19}$$

For the boundary values of the vector potential, we find:

$$\begin{aligned}\tilde{A}_{z,i} &= \tilde{A}_z(r_i) = \frac{\mu}{p}\varepsilon\left(r_i\tilde{H}_{\varphi,i} - kr_o\tilde{H}_{\varphi,o}\right)\\ \tilde{A}_{z,o} &= \tilde{A}_z(r_o) = \frac{\mu}{p}\varepsilon\left(kr_i\tilde{H}_{\varphi,i} - r_o\tilde{H}_{\varphi,o}\right)\end{aligned} \tag{20}$$

We can find the values of $\tilde{H}_\varphi$ for each border from the set of equations consisting of the boundary conditions (15) and (16) for all borders and the characteristic equations (20) for all regions.
We cannot use expression (18) for field computations in ideal iron, because in that case $\mu = \infty$ and $H_\varphi = 0$ are valid. For that reason, we express the coefficients C and D as functions of the vector potential $\tilde{A}_z$ instead of the magnetic field strength $\tilde{H}_\varphi$. In this way we get

$$\tilde{A}_z(r) = \frac{k\varepsilon}{2}\left(\tilde{A}_{z,i}\left[\frac{r_o^p}{r^p} - \frac{r^p}{r_o^p}\right] + \tilde{A}_{z,o}\left[\frac{r^p}{r_i^p} - \frac{r_i^p}{r^p}\right]\right) \tag{21}$$

4. - **A MAGNETIC NETWORK**

The quantities used in the set of equations (15), (16) and (20) ($\tilde{H}_\varphi$ and $\tilde{A}_z$) are not directly accessible from an electric circuit. When we use a magnetic network, we get a better relation with electric quantities. For this purpose, we use the pole flux (using (12))

$$\Phi(r) = l\int_{-\frac{\pi}{2p}}^{\frac{\pi}{2p}} B_r(r,\varphi)r\,\mathrm{d}\varphi = 2l\tilde{A}_z(r) \tag{22}$$

where l is the length of the magnetic part of the machine. Using this, we can write the boundary condition (15) as

$$\Phi_{o,m} = \Phi_{i,m+1} \tag{23}$$

We can see this equation as the node equation of a network representation in which the fluxes are seen as currents.
Further, we introduce the magnetic "voltage" (using (13))

$$U_m(r) = \int_0^{\pi/p} H_\varphi(r,\varphi)r\,\mathrm{d}\varphi = \frac{2r}{p}\tilde{H}_\varphi(r) \tag{24}$$

Here, subscript m stands for magnetic, and does not denote the region.
Using (24), we can write the boundary condition (16) as

$$-U_{m,o,m} + U_{m,i,m+1} = F_{m,m} \tag{25}$$

where we introduced the magnetomotive force

$$F_{m,m} = \frac{2r_m}{p} i_m \hat{Z}_m = i_m \frac{N_m}{p} \tag{26}$$

which corresponds with the maximum of the enclosed current for one pole pitch (N_m/p is the number of conductors of one pole pitch). The first subscript of $F_{m,m}$ stands for magnetic and the second denotes the border number. The radius r_m is the radius of the border between the regions m and $m+1$: $r_m = r_{o,m} = r_{i,m+1}$.
Using (22) and (24), we find for the boundary values ((20)):

$$\begin{aligned} \Phi_i &= \mu l \varepsilon \left(U_{m,i} - kU_{m,o}\right) = \frac{U_{m,i}}{R_{m\sigma}} + \frac{U_{m,i} - U_{m,o}}{R_{mm}} \\ \Phi_o &= \mu l \varepsilon \left(kU_{m,o} - U_{m,o}\right) = -\frac{U_{m,o}}{R_{m\sigma}} + \frac{U_{m,i} - U_{m,o}}{R_{mm}} \end{aligned} \tag{27}$$

where we introduced

$$R_{mm} = \frac{1}{\mu l \varepsilon k} \quad ; \quad R_{m\sigma} = \frac{1}{\mu l \varepsilon (1-k)} \tag{28}$$

Here, the first letter in the subscripts stands for magnetic and the second denotes the main flux, respectively the leakage flux.
We can see (27) as the set of equations representing a two-port (figure 3). So, we can see each cylindrical region as a two-port and the whole machine as a ladder network of two-ports, connected according to the boundary conditions (23) and (25).

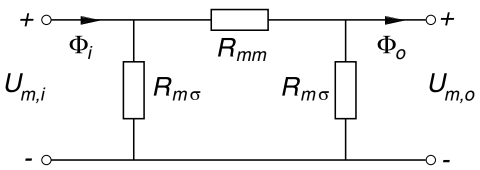

Figure 3 The magnetic network for a region

We can find the values of U_m from the boundary conditions (23) and (25) and the characteristic equation for the region (27) or by the network representation (figure 3).

5. - **AN EQUIVALENT CIRCUIT**

We can get a better relation between the boundary quantities U_m and Φ and the electric quantities, when we convert the magnetic network into an equivalent circuit ([5]).
For this purpose we replace the magnetic voltages $U_{m,i}$ and $U_{m,o}$ by, respectively, the currents i_i and i_o. These currents are not real currents, but currents representing the magnetic voltages:

$$i_i = U_{m,i} \quad ; \quad i_o = U_{m,o} \tag{29}$$

Further, we replace the flux Φ by the flux linkage λ.
Before this replacement, we consider a sinusoidally distributed, infinitely thin winding at radius r_0 with the magnetic axis in the direction $\varphi = \alpha_0/p$:

$$Z_0(r_0,\varphi) = \frac{N_0}{2r_0} \sin\left(p\varphi - \alpha_0\right) \tag{30}$$

The flux linked with this winding may be found by

$$\lambda(r_0) = l \int_0^{2\pi} A_z(r_0,\varphi) Z_0(r_0,\varphi) r_0 \,\mathrm{d}\varphi \tag{31}$$

Substituting (30), (10), and (11) with (22) into this expression leads to

$$\lambda_0 = \frac{\pi}{4} N_0 \Phi_0 \cos\alpha_0 \tag{32}$$

It should be noted that the factor $\pi/4$ is the winding factor of a sinusoidally distributed winding.
Now, we introduce the flux linkages which replace the fluxes Φ_i and Φ_o:

$$\lambda_i = \frac{\pi}{4}\Phi_i \quad ; \quad \lambda_o = \frac{\pi}{4}\Phi_o \tag{33}$$

In fact, these are the flux linkages according to (32) for the case that $N_0 = 1$ and $\alpha_0 = 0$.
Using (29) and (33), the boundary conditions (23) and (25) become:

$$\lambda_{o,m} = \lambda_{i,m+1} \quad ; \quad -i_{o,m} + i_{i,m+1} = F_{m,m} \tag{34}$$

Using (29) and (33), we can use the set of network equations (27) to create the equivalent circuit in figure 4, in which we introduced

$$L_m = \frac{\pi}{4}\frac{1}{R_{mm}} = k\frac{\pi}{4}\mu l \varepsilon \quad ; \quad L_\sigma = \frac{\pi}{4}\frac{1}{R_{m\sigma}} = (1-k)\frac{\pi}{4}\mu l \varepsilon \tag{35}$$

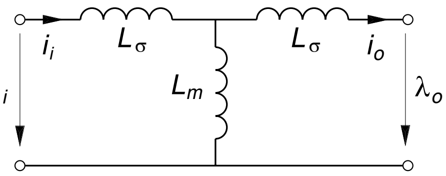

Figure 4 The equivalent circuit for a region

When we consider the expressions (35) and figure 4, we can see that the parameter k in fact is the coupling factor.
For the computation of the field in the region, we can use the vector potential ((21) with (22) and (33)):

$$\tilde{A}_z(r) = \frac{4}{\pi}\frac{k\varepsilon}{4l}\left(\lambda_i \left[\frac{r_o^p}{r^p} - \frac{r^p}{r_o^p}\right] + \lambda_o \left[\frac{r^p}{r_i^p} - \frac{r_i^p}{r^p}\right]\right) \tag{36}$$

6. - **THE BASICS OF THE MACHINE MODEL**

Before deriving the machine-model equations for the case of three windings on the stator and three windings on the rotor, we first consider the simpler case of two windings on the horizontal axis in this section.
Here, we only excite stator winding *sa* and rotor winding *ra*, while the magnetic axis of last winding is along the horizontal axis ($\theta = 0$, see figure 2). So, there only is a surface current density at $r = r_2 = r_r$ and at $r = r_3 = r_s$. In fact, we now have a transformer with those windings.
As we can see in figure 1, our induction machine model consists of 5 regions, so that we have to place 5 equivalent

circuits according to figure 4 in cascade. However, in the second and the fourth region (r and s), the relative permeability is assumed to be infinite. So, the inductances in figure 4 are infinite too. This means that there are no currents in the corresponding equivalent circuits and we get three separate regions (the first, third, and fifth region). Because there is no excitation in the first (i) and the fifth (e) region, there are also no currents in these regions. The only active equivalent circuit is the third region, the air gap (δ).
For the terminals of this equivalent circuit, we use the boundary equation for the currents in (34) (using (26) for F_m):

$$-i_{o,2}+i_{i,3}=F_{m,2}=i_2\frac{N_2}{p} \quad ; \quad -i_{o,3}+i_{i,4}=F_{m,3}=i_3\frac{N_3}{p}$$

Here, $i_{o,2}$ and $i_{i,4}$ are zero. Further, we replace the subscript 2 by ra (rotor phase a) and the subscript 3 by sa (stator phase a) with F_m and leave out the subscript 3 with the currents in the equivalent circuit:

$$i_i=F_{m,ra}=i_{ra}\frac{N_r}{p} \quad ; \quad -i_o=F_{m,sa}=i_{sa}\frac{N_s}{p} \tag{37}$$

When we compare the equations for λ (32) and (33), we can see that we have to multiply λ_i and λ_o by the number of turns of a winding to find the flux linkage of this winding. In this way we get for the coils corresponding with one pole pair (all windings are connected in series):

$$\lambda_{rac}=\frac{N_r}{p}\lambda_i \quad ; \quad \lambda_{sac}=\frac{N_s}{p}\lambda_o \tag{38}$$

The equations (37) and (38) correspond with two ideal transformers. So, we get the circuit in figure 5.

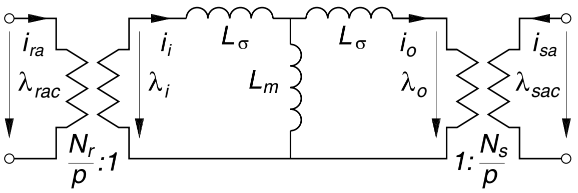

Figure 5 An equivalent circuit for the basic model

To compare this figure with its classical form, we investigate the main inductance. Using (35) with (19), we get

$$L_m=\frac{\pi}{4}\frac{1}{R_{mm}} \quad ; \quad R_{mm}=\frac{1}{2\mu_0 l}\left[\frac{r_s^p}{r_r^p}-\frac{r_r^p}{r_s^p}\right] \tag{39}$$

The expression for the main reluctance R_{mm} may not look very familiar. However when we work it out for the case that the air gap is relatively small, we will find an usual expression. For this purpose we use the relation

$$r_s=r_r+\delta \tag{40}$$

where δ is the air-gap length. When the air gap is relatively small, the expression for R_{mm} in (39) becomes

$$R_{mm}=\frac{1}{\mu_0 2l}\delta\frac{2p}{r_s}=2\frac{\delta}{\mu_0\left(\frac{2}{\pi}\tau_p l\right)} \quad ; \quad \delta \ll r_s \tag{41}$$

where $\tau_p=\pi r_s/p$ is the pole pitch. As we may see, R_{mm} is twice the reluctance of the air-gap over one pole pitch. The factor $2/\pi$ accounts for averaging the sinusoidal distribution of the flux density.

7. - **THE MACHINE MODEL**

In this section we derive the usual machine-model equations, using the previous sections. We start with the computation of the air-gap flux. Next, the flux linkages of the machine windings are evaluated. These flux linkages are used in the voltage equations, which are transformed to the stator reference frame.

In the previous section, on the stator, we only used phase a. When we use all three stator phases, the surface current density becomes (corresponding with (3)):

$$\begin{aligned}K_s(\varphi)=\hat{Z}_s\big[&i_{sa}\sin p\varphi+i_{sb}\sin(p\varphi-\tfrac{2}{3}\pi)\\&+i_{sc}\sin(p\varphi-\tfrac{4}{3}\pi)\big]\end{aligned} \tag{42}$$

The different magnetic axes are depicted in figure 2. The rotor surface current density is

$$\begin{aligned}K_r(\varphi)=\hat{Z}_r\big[&i_{ra}\sin(p\varphi-p\theta)+i_{rb}\sin\left(p\varphi-p\theta-\tfrac{2}{3}\pi\right)\\&+i_{rc}\sin\left(p\varphi-p\theta-\tfrac{4}{3}\pi\right)\big]\end{aligned} \tag{43}$$

A three-phase sinusoidally distributed current density may also be represented by two orthogonal sinusoidally distributed current densities. To obtain these current densities, we may use the well-known transformations named after Clarke and Park:

$$\begin{aligned}\begin{bmatrix}f_{s\alpha}\\f_{s\beta}\end{bmatrix}&=\frac{2}{3}\begin{bmatrix}1 & -\frac{1}{2} & -\frac{1}{2}\\0 & \frac{1}{2}\sqrt{3} & -\frac{1}{2}\sqrt{3}\end{bmatrix}\begin{bmatrix}f_{sa}\\f_{sb}\\f_{sc}\end{bmatrix}\\\begin{bmatrix}f_{r\alpha}\\f_{r\beta}\end{bmatrix}&=\frac{2}{3}\begin{bmatrix}\cos p\theta & \cos(p\theta+\frac{2}{3}\pi) & \cos(p\theta+\frac{4}{3}\pi)\\\sin p\theta & \sin(p\theta+\frac{2}{3}\pi) & \sin(p\theta+\frac{4}{3}\pi)\end{bmatrix}\begin{bmatrix}f_{ra}\\f_{rb}\\f_{rc}\end{bmatrix}\end{aligned} \tag{44}$$

When we apply these transformations, the machine currents are represented in the stator reference frame. Using the transformed currents, the current densities ((42) and (43)) may also be given as

$$\begin{aligned}K_s(\varphi)&=\tfrac{3}{2}\hat{Z}_s\left(i_{s\alpha}\sin p\varphi-i_{s\beta}\cos p\varphi\right)\\K_r(\varphi)&=\tfrac{3}{2}\hat{Z}_r\left(i_{r\alpha}\sin p\varphi-i_{r\beta}\cos p\varphi\right)\end{aligned} \tag{45}$$

The sine wave in this expression corresponds with the magnetic axis in the horizontal (α) direction, the cosine wave with the magnetic axis in the vertical (β) direction (see figure 2).
In the sections 2 to 6, we used the surface current density $K_m(\varphi)=i_m\hat{Z}_m\sin p\varphi$. If we compare this expression for $K_m(\varphi)$ with (45), we can see that we have to apply a factor $3/2$ for the currents. Further, we may use all derived equations. However, we now have two magnetic axes (α and β). So the equations for the currents (magnetomotive forces) (37) become

$$\begin{bmatrix}i_{i\alpha}\\i_{i\beta}\end{bmatrix}=\frac{3}{2}\frac{N_r}{p}\begin{bmatrix}i_{r\alpha}\\i_{r\beta}\end{bmatrix} \quad ; \quad \begin{bmatrix}i_{o\alpha}\\i_{o\beta}\end{bmatrix}=-\frac{3}{2}\frac{N_s}{p}\begin{bmatrix}i_{s\alpha}\\i_{s\beta}\end{bmatrix}$$

With these expressions, we find for the equivalent-circuit flux linkages (see figure 4):

$$\begin{bmatrix}\lambda_{o\alpha}\\ \lambda_{o\beta}\end{bmatrix} = L_\sigma \frac{3}{2}\frac{N_s}{p}\begin{bmatrix}i_{s\alpha}\\ i_{s\beta}\end{bmatrix} + L_m \frac{3}{2}\frac{N_s}{p}\left(\begin{bmatrix}i_{s\alpha}\\ i_{s\beta}\end{bmatrix} + n_{rs}\begin{bmatrix}i_{r\alpha}\\ i_{r\beta}\end{bmatrix}\right)$$
$$\begin{bmatrix}\lambda_{i\alpha}\\ \lambda_{i\beta}\end{bmatrix} = L_\sigma \frac{3}{2}\frac{N_s}{p}\begin{bmatrix}i_{r\alpha}\\ i_{r\beta}\end{bmatrix} + L_m \frac{3}{2}\frac{N_s}{p}\left(\begin{bmatrix}i_{s\alpha}\\ i_{s\beta}\end{bmatrix} + n_{rs}\begin{bmatrix}i_{r\alpha}\\ i_{r\beta}\end{bmatrix}\right) \tag{46}$$

where we used $n_{rs} = N_r/N_s$

We can find the flux linked with each of the stator and rotor windings by means of (32) with (33), where the angle α (the angle between the magnetic axis of the field and the magnetic axis of the winding) follows from figure 2:

$$\begin{bmatrix}\lambda_{sa}\\ \lambda_{sb}\\ \lambda_{sc}\end{bmatrix} = N_s \begin{bmatrix}1 & 0\\ -\frac{1}{2} & \frac{1}{2}\sqrt{3}\\ -\frac{1}{2} & -\frac{1}{2}\sqrt{3}\end{bmatrix}\begin{bmatrix}\lambda_{o\alpha}\\ \lambda_{o\beta}\end{bmatrix}$$
$$\begin{bmatrix}\lambda_{ra}\\ \lambda_{rb}\\ \lambda_{rc}\end{bmatrix} = N_r \begin{bmatrix}\cos p\theta & \sin p\theta\\ \cos(p\theta+\frac{2}{3}\pi) & \sin(p\theta+\frac{2}{3}\pi)\\ \cos(p\theta+\frac{4}{3}\pi) & \sin(p\theta+\frac{4}{3}\pi)\end{bmatrix}\begin{bmatrix}\lambda_{i\alpha}\\ \lambda_{i\beta}\end{bmatrix} \tag{47}$$

In fact, we introduced the inverse transformations of (44) in a natural way.
When we add the stator and the rotor phase-winding voltage equations, which have the form $u = Ri + \frac{\mathrm{d}}{\mathrm{d}t}\lambda$, we have a complete set of equations for the induction machine.
To get a more practical (and usual) form of the voltage equations, we use the transformations (44) for the phase-winding voltage equations:

$$\begin{bmatrix}u_{s\alpha}\\ u_{s\beta}\end{bmatrix} = R_s\begin{bmatrix}i_{s\alpha}\\ i_{s\beta}\end{bmatrix} + \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\lambda_{s\alpha}\\ \lambda_{s\beta}\end{bmatrix}$$
$$\begin{bmatrix}u_{r\alpha}\\ u_{r\beta}\end{bmatrix} = R_r\begin{bmatrix}i_{r\alpha}\\ i_{r\beta}\end{bmatrix} + \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\lambda_{r\alpha}\\ \lambda_{r\beta}\end{bmatrix} + p\omega_m\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\lambda_{r\alpha}\\ \lambda_{r\beta}\end{bmatrix} \tag{48}$$

and for the equations for the flux linkages (47):

$$\begin{bmatrix}\lambda_{s\alpha}\\ \lambda_{s\beta}\end{bmatrix} = N_s\begin{bmatrix}\lambda_{o\alpha}\\ \lambda_{o\beta}\end{bmatrix} \quad ; \quad \begin{bmatrix}\lambda_{r\alpha}\\ \lambda_{r\beta}\end{bmatrix} = N_r\begin{bmatrix}\lambda_{i\alpha}\\ \lambda_{i\beta}\end{bmatrix} \tag{49}$$

Using (46), these equations may be elaborated to

$$\begin{bmatrix}\lambda_{s\alpha}\\ \lambda_{s\beta}\end{bmatrix} = L_{s\sigma}\begin{bmatrix}i_{s\alpha}\\ i_{s\beta}\end{bmatrix} + L_{sm}\left(\begin{bmatrix}i_{s\alpha}\\ i_{s\beta}\end{bmatrix} + n_{rs}\begin{bmatrix}i_{r\alpha}\\ i_{r\beta}\end{bmatrix}\right)$$
$$\begin{bmatrix}\lambda_{r\alpha}\\ \lambda_{r\beta}\end{bmatrix} = L_{s\sigma}n_{rs}^2\begin{bmatrix}i_{r\alpha}\\ i_{r\beta}\end{bmatrix} + n_{rs}L_{sm}\left(\begin{bmatrix}i_{s\alpha}\\ i_{s\beta}\end{bmatrix} + n_{rs}\begin{bmatrix}i_{r\alpha}\\ i_{r\beta}\end{bmatrix}\right) \tag{50}$$

where we introduced

$$L_{s\sigma} = \frac{3}{2}\frac{N_s^2}{p}L_\sigma \quad ; \quad L_{sm} = \frac{3}{2}\frac{N_s^2}{p}L_m \tag{51}$$

The set of equations (48) and (50) is the usual set of equations for induction machines.
The leakage inductances found in this way only incorporate the leakage flux in the air gap, which is normally very small. However, they can simply be enlarged to take into account the other contributions to the leakage flux, which are not in the model here.

8. - EXAMPLES

When we have found the flux linkages, we may use (10) with (36) to compute the vector potential in the rotor,the air gap and the stator. We can use this vector potential to draw (analytically computed) field plots in an easy way.
As an example, figure 6 shows the bending of the field lines in the air gap corresponding to the torque for normal motor operation. Figure 7 makes the air-gap leakage flux visible for the case of a high slip. In both cases, the inner region in figure 1 (*i*) is left out.
In these (teaching) examples the air-gap width is enlarged to make the interesting phenomena very clear.

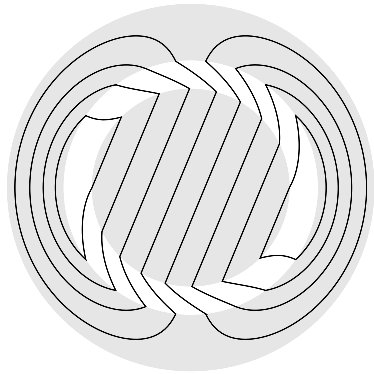

Figure 6 Low-slip operation

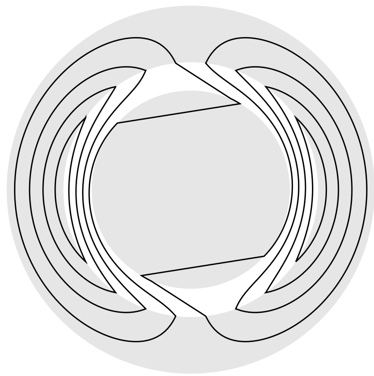

Figure 7 High-slip operation

9. - REFERENCES

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